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THE CONDITIONS FOR THE NON-LINEAR STABILITY OF PLANE AND HELICAL MHD FLOWS†

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The stability of the steady flows of an ideal incompressible fluid of uniform density in a magnetic field is investigated. Only those MHD flows are considered which possess one of the types of symmetry (translational, axial, rotational or helical). The sufficient conditions for non-linear stability of the flows in question with respect to perturbations of this symmetry are obtained. These conditions are proved by the method of coupling the integrals of motion [1, 2] in the form [3–8], based on constructing functionals having absolute minima on specified steady solutions. Each of the functionals constructed is the sum of the kinetic energy, the integral of an arbitrary function of the Lagrangian coordinate and another integral, specific for the flows being investigated. The use of Lagrangian coordinate fields leads to a whole family of new definitions of stability. According to these definitions, deviations of the perturbed flows from the unperturbed ones are measured by the integrals of the squares of the velocity-field and Lagrangian-coordinate perturbations. The stability conditions obtained are extended to existing results [5–7, 9] on new types of flows. These conditions are of an a priori nature since the corresponding theorems of existence of the solutions are not proved.

1. FORMULATION OF THE PROBLEM

Three-dimensional motions of an ideal incompressible fluid containing a magnetic field $\mathbf{h} = (h_1, h_2, h_3)$ in the region τ with a fixed solid ideally conducting boundary $\partial\tau$, are considered. The equations describing such motions will be taken in the form [10]

$$\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} = -\nabla p - (4\pi)^{-1}[\mathbf{h}\operatorname{rot}\mathbf{h}], \qquad \mathbf{h}_t = \operatorname{rot}[\mathbf{uh}], \ \operatorname{div}\mathbf{h} = 0, \ \operatorname{div}\mathbf{u} = 0 \tag{1.1}$$

We will assume that the following conditions are satisfied on the boundary $\partial\tau$

$$\mathbf{un} = 0, \quad \mathbf{hn} = 0 \tag{1.2}$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is the unit normal to $\partial\tau$. The initial data for the system of equations (1.1) are specified in the form

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{h}(\mathbf{x}, 0) = \mathbf{h}_0(\mathbf{x}) \tag{1.3}$$

so that the functions $\mathbf{u}_0(x)$ and $\mathbf{h}_0(x)$ are solenoidal everywhere in the region and satisfy conditions (1.2) on its boundary $\partial\tau$. All the functions used together with their derivatives, which occur in the equation of motion (1.1), are assumed to be continuous.

In the last sections we will investigate the stability of some special exact steady solutions of problem (1.1)–(1.3) with respect to special classes of perturbations.

2. FLOWS WITH HELICAL SYMMETRY

We will investigate motions, all the fields in which depend on r, $\mu = a\varphi - bz$ and t (here a is any integer and b is any real number) in a cylindrical system of coordinates r, φ, z. We will assume that the magnetic field $\mathbf{h}$ has only an angular component h_2 and an axial component h_3, connected with one another by the relation

$$ah_2 - brh_3 = 0 \tag{2.1}$$

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Using the notation [6, 7]

$$\lambda = a\upsilon - brw, \quad \beta = aw + br\upsilon, \quad R = a^2 + b^2 r^2, \quad K = 2abR^{-2}$$

$$p^* = p + (8\pi)^{-1}\left(h_2^2 + h_3^2\right), \quad p_1 = \beta^2, \quad g_1 = b^2 r R^{-2}$$

$$\rho_2 = (4\pi)^{-1}(h_2 / r)^2, \quad g_2 = -r, \quad \rho_3 = h_3$$

and condition (2.1), the equations of motion (1.1) can be converted to the form

$$Du - K\beta\lambda - r^{-1}(a\lambda)^2 R^{-2} = -p_r^* + \rho_1 g_1 + \rho_2 g_2 \tag{2.2}$$

$$D(r\lambda R^{-1}) + K\beta r u = -p_\mu^*, \quad D\rho_1 = 0, \quad D\rho_2 = 0, \quad Dp_3 = 0$$

$$u_r + u / r + r^{-1}\lambda_\mu = 0, \quad D = \partial / \partial t + u\partial / \partial r + r^{-1}\lambda\partial / \partial\mu$$

We will introduce the additional scalar function $q(r, \mu, t)$, the values of which remain the same in each fluid particle [8]

$$Dq = 0 \tag{2.3}$$

Since the motions being investigated occur in a fixed region, its boundaries must possess the required symmetry, i.e. they are specified by functions of two variables (α is the number of the boundary component)

$$s_\alpha(r, \mu) = 0 \tag{2.4}$$

Boundary conditions (1.2) then take the form

$$u\left(s_\alpha\right)_r + (\lambda / r)\left(S_\alpha\right)_\mu = 0 \tag{2.5}$$

We will assume that the region τ of the flow is a doubly connected region ($\alpha = 1, 2$), and its boundary $\partial\tau$ (2.4) consists of two components: internal and external.

The initial data (1.3) for Eqs (2.2) and (2.3) will be written as follows:

$$u(r,\mu,0) = u_0(r,\mu), \quad \lambda(r,\mu,0) = \lambda_0(r,\mu) \tag{2.6}$$

$$\rho_1(r,\mu,0) = \rho_{10}(r,\mu), \quad \rho_2(r,\mu,0) = \rho_{20}(r,\mu)$$

$$\rho_3(r,\mu,0) = \rho_{30}(r,\mu), \quad q(r,\mu,0) = q_0(r,\mu)$$

It should be noted that relation (2.1) is a consequence of the second equation of system (1.1). In fact, we can obtain the following relation from this equation by simple reduction

$$D\left((a / r)h_2 - bh_3\right) = 0$$

which shows that if we choose the initial components of the magnetic field $h_{20}(r, \mu)$ and $h_{30}(r, \mu)$, which satisfy condition (2.1), this condition will hold not only when $t = 0$ but also at any subsequent instant of time.

The following energy integral holds for problem (2.1)–(2.6).

$$\mathrm{E} = T + \Pi_1 + \Pi_2 = \text{const}, \quad T = \frac{1}{2}\int_\tau \left(\lambda^2 R^{-1} + u^2\right)d\tau, \quad d\tau = r dr d\mu \tag{2.7}$$

$$\Pi_1 = \int_\tau \rho_1 U_1 d\tau, \quad U_1 = U_1(r) = (2R)^{-1} + C_1, \quad \Pi_2 = \int_\tau \rho_2 U_2 d\tau, \quad U_2 = U_2(r) = r^2 / 2 + C_2$$

where C_1 and C_2 are constants representing the values of the functions U_1 and U_2 either on the internal or external part of the boundary $\partial\tau$ (2.4). Another integral of this problem is defined in terms of the arbitrary function $\Phi(q)$

$$I = \int_\tau \Phi(q)d\tau = \text{const} \tag{2.8}$$

Problem (2.1)–(2.6) has exact steady solutions

$$\lambda = u = 0; \quad \rho_i = \rho_i^0(r), \quad i = 1,2,3; \quad q = Q(r) \tag{2.9}$$

in which $\rho_i^0(r)$ and $Q(r)$ are arbitrary functions of argument r. If $dQ/dr \neq 0$ in τ [11], we have from relations (2.7) and (2.9)

$$\rho_i^0 = \rho_i^0(Q), \quad U_1 = U_1(Q), \quad U_2 = U_2(Q) \tag{2.10}$$

$$Q \in \left(Q^-, Q^+\right), \quad Q^- = \min Q(r), \quad Q^+ = \max Q(r) \text{ in } \tau$$

The exact unsteady solutions of problem (2.1)–(2.6) can be written in the form

$$u = u(r,\mu,t), \quad \lambda = \lambda(r,\mu,t), \quad \rho_i = \rho_i^0(Q) + \sigma_i(r,\mu,t)$$

$$q = Q(r) + \kappa(r,\mu,t)$$

where u, λ, σ_i, κ are regarded as perturbations of solution (2.9) and (2.10). We will assume that, when the initial data (2.6) are appropriately specified, such solutions exist and are continuous, and the derivatives in the equations of motion (2.2) and (2.3) are also continuous. Using (2.10) we can construct the additional function of two variables $V = V(X, Y)$

$$V = \rho_1^{0'}(X)\left[U_1(Y) - U_1(X)\right] + \rho_2^{0'}(X)\left[U_2(Y) - U_2(X)\right], \quad X, Y \in \left(Q^-, Q^+\right) \tag{2.11}$$

Here and below the prime denotes a derivative with respect to the argument.

Assertion 1. Suppose the following inequality is satisfied over the whole region τ

$$0 \leqslant c_1^- \leqslant \partial V / \partial X \leqslant c_1^+ < +\infty \tag{2.12}$$

with regard to the constants c_1^- and c_1^+. Then, at any instant of time, the perturbations u, λ and κ can be estimated in terms of their initial values u_*, λ_* and k_* as follows:

$$\int_\tau \left(\lambda^2 R^{-1} + u^2 + c_1^{-1}\kappa^2\right)d\tau \leqslant \int_\tau \left(\lambda_*^2 R^{-1} + u_*^2 + c_1^+\kappa_*^2\right)d\tau \tag{2.13}$$

This can be proved by the method of coupling the integrals of motion (2.7) and (2.8) [1–8].

Suppose the initial fields $\rho_{10}(r, \mu)$, $\rho_{20}(r, \mu)$ and $q_0(r, \mu)$ (2.6) can be obtained from the steady distributions $\rho_1^0(r)$, $\rho_2^0(r)$ and $Q(r)$ given by (2.9) solely by permutations of the particles of the incompressible fluid. The quantities ρ_1, ρ_2 and q during these permutations are constant in each fluid particle and are equal to their values assumed in the solutions (2.9)

$$\rho_1 = \rho_1^0(q), \quad \rho_2 = \rho_2^0(q), \quad q \in \left(Q^-, Q^+\right)$$

$$\rho_1 \in \left(\rho_1^{0-}, \rho_1^{0+}\right), \quad \rho_1^{0-} = \min \rho_1^0(r), \quad \rho_1^{0+} = \max \rho_1^0(r) \text{ in } \tau \tag{2.14}$$

$$\rho_2 \in \left(\rho_2^{0-}, \rho_2^{0+}\right), \quad \rho_2^{0-} = \min \rho_2^0(r), \quad \rho_2^{0+} = \max \rho_2^0(r) \text{ in } \tau$$

i.e. the relations between ρ_1, ρ_2 and q, and the ranges within which these quantities are defined are the same as in (2.1). Here we are essentially dealing with the "equivorticity" condition [12]. Clearly, if relations (2.14) hold at the initial instant they will remain the same at any subsequent instant.

From integrals (2.7) and (2.8) we set up the conserving functional

$$F = F(u, \lambda, q) = \frac{1}{2}\int_\tau \left\{\lambda^2 R^{-1} + u^2 + 2\rho_1(q)U_1(Q) + 2\rho_2(q)U_2(Q) + 2\Phi(q)\right\} d\tau$$

which we will represent in the form of the sum of three terms

$$\begin{gathered} F = F(u, \lambda, q) = F(0, 0, Q) + F_1 + F_2 \\ F_1 = \int_\tau \left\{\kappa f_q(Q, Q)\right\} d\tau, \quad F_2 = \frac{1}{2}\int_\tau \left\{\lambda^2 R^{-1} + u^2 + 2 f_1(q, Q)\right\} d\tau \\ f_1(q, Q) = f(q, Q) - f(Q, Q) - \kappa f_q(Q, Q) \\ f(q, Q) = \Phi(q) + U_1(Q)\rho_1(q) + U_2(Q)\rho_2(q), \quad f_q(q, Q) = \frac{\partial f(q, Q)}{\partial q} \end{gathered} \tag{2.15}$$

Using the fact that the function $\Phi(q)$ is arbitrary, we can choose it so that $f_q(Q, Q) = 0$, namely

$$\Phi'(Q) = -U_1(Q)\rho_1^{0\prime}(Q) - U_2(Q)\rho_2^{0\prime}(Q) \tag{2.16}$$

It then follows from (2.15) that $F_1 = 0$, while the functional F_2 is independent of time. By virtue of (2.11) and (2.16) we obtain for the function V

$$V(q, Q) = f_q(q, Q) \tag{2.17}$$

Relations (2.11) and (2.17) give the inequalities

$$0 \leqslant c_1^- \leqslant f_{qq}(q, Q) \leqslant c_1^+ < +\infty \tag{2.18}$$

which denote that f is convex in the interval (Q^-, Q^+), i.e. over the whole range of values of the argument q. Taking the formula for the residual term in Lagrangian form [13], the function f_1 given by (2.15) can be converted to the form

$$f_1(q, Q) = \tfrac{1}{2} f_{qq}(Q_*, Q)\kappa^2, \quad Q_* = Q + \vartheta\kappa, \quad 0 < \vartheta < 1, \quad \kappa = q - Q$$

We can therefore rewrite (2.18) as follows:

$$c_1^-\kappa^2 / 2 \leqslant f_1(q, Q) \leqslant c_1^+\kappa^2 / 2 \tag{2.19}$$

whence, since the functional F_2 given by (2.15) is constant in time, we obtain the estimate (2.13).

Suppose now that the initial field of the Lagrangian coordinate $q_0(r, \mu)$ is arbitrary, while the initial fields $\rho_{10}(r, \mu)$, $\rho_{20}(r, \mu)$ are calculated from it using (2.14). In this case it becomes necessary to consider values of q which lie outside the interval (Q^-, Q^+). To do this, the definition of the function $f(q, Q)$ given by (2.15) is supplemented over the whole q-axis while preserving inequalities (2.19). Using the three functions $\Phi(q)$, $\rho_1^0(q)$ and $\rho_2^0(q)$, which are arbitrary outside (Q^-, Q^+) this extension can clearly be carried out by an infinite number of methods. The required estimate (2.13) again follows from (2.19), where $-\infty < q < +\infty$, and the condition for F_2 to be independent of time. Assertion 1 is proved.

The problem of the non-linear stability of steady helical flows of an ideal incompressible fluid of uniform density when there is no magnetic field was considered previously in [6, 7]. The sufficient condition for the stability of these flows to finite perturbations of the same type of symmetry was obtained in the form

$$0 \leqslant c_3^- \leqslant g_1 / \left(\rho_1^0\right)_r \leqslant c_3^+ < +\infty \tag{2.20}$$

where c_3^- and c_3^+ are constants. A comparison of (2.12) and (2.20) shows that the second of these is a special case of the first if $\mathbf{h} = 0$, $q = \rho_1$ (2.2).

Hence, for $q \neq \rho_1$, $\mathbf{h} = 0$ Assertion 1 leads to criteria of the stability of helical flows of an ideal fluid within the framework of new definitions which differ from that proposed in [6, 7]. In addition, when $h \neq 0$, Assertion 1 must be treated as an extension of the result obtained in [6, 7] on MHD flow.

When $a = 0$ all the results obtained in this section can be transferred directly to rotationally symmetric motion.

3. PLANE FLOWS

We will investigate, in Cartesian coordinates x, y, z, motions, all the fields in which are independent of the z coordinate, where the velocity field $\mathbf{u} = \mathbf{u}(x, y, t) = (u, \upsilon, 0)$ while the magnetic field $\mathbf{h} = \mathbf{h}(x, y, t) = (0, 0, h_3)$. The governing equations of motion (1.1) then take the form

$$Du = -p^*_x, \quad D\upsilon = -p^*_y, \quad Dh_3 = 0, \quad u_x + \upsilon_y = 0 \tag{3.1}$$

$$D = \partial / \partial t + u\partial / \partial x + \upsilon\partial / \partial y$$

where $p^* = p + (8\pi)^{-1}h_3^2$ is the modified pressure. We will assume that the fluid moves within a fixed region τ, the components of the boundary of which have the form of cylindrical surfaces with generatrices parallel to the z-axis and defined by the relations

$$S_\alpha(x, y) = 0, \quad \alpha = 1, 2, \ldots, m \tag{3.2}$$

Here α is the number of the components of the boundary of the region. In the x, y plane the region τ of the flow is bounded by the curves $\partial\tau_\alpha$ (3.2), so that the following relation holds for the boundary $\partial\tau$

$$\partial\tau = \bigcup_{\alpha=1}^{m} \partial\tau_\alpha \tag{3.3}$$

The non-flow boundary conditions (1.2) on $\partial\tau$ (3.3) give

$$u(s_\alpha)_x + \upsilon(s_\alpha)_y = 0 \tag{3.4}$$

The initial data (1.3) for Eqs (3.1) become

$$u(x, y, 0) = u_0(x, y), \quad \upsilon(x, y, 0) = \upsilon_0(x, y) \tag{3.5}$$

$$h_3(x, y, 0) = h_{30}(x, y)$$

We will introduce the stream function ψ and the vortex field ω such that

$$u = -\psi_y, \quad \upsilon = \psi_x, \quad \omega = \upsilon_x - u_y = \Delta\psi \tag{3.6}$$

Eliminating the modified pressure p^* in the first two equations of system (3.1) using (3.6) we obtain the vortex transfer equation

$$D\omega = 0 \tag{3.7}$$

It follows from (3.1) and (3.7) that both the magnetic field and the vorticity are conserved in each fluid particle. Consequently, we can consider an arbitrary scalar field $q(x, y, t)$ which is also transferred by each of the fluid particles when the fluid moves [8]

$$Dq = 0 \tag{3.8}$$

In this connection we will investigate below problem (3.1)–(3.4), (3.7) in which the third of Eqs (3.1) is replaced by Eq. (3.8). Then the initial data (3.5) will be written in the form

$$\psi(x, y, 0) = \psi_0(x, y), \quad q(x, y, 0) = q_0(x, y) \tag{3.9}$$

The energy integral for problem (3.1)–(3.4), (3.7)–(3.9) then takes the following form

$$2E = \int_\tau (u^2 + v^2) d\tau = \text{const}, \quad d\tau = dxdy \tag{3.10}$$

By virtue of (3.8) we have the integral of motion

$$I = \int_{\tau} \Phi(q)d\tau = \text{const} \tag{3.11}$$

where $\Phi(q)$ is an arbitrary function of argument q. Moreover, the quantities

$$\Gamma_\alpha = \int_{\partial\tau_\alpha} (\mathbf{n}\nabla\psi)dl = \text{const}, \quad dl = \left(dx^2 + dy^2\right)^{1/2} \tag{3.12}$$

which have the meaning of the circulations of the velocity over the contours of the boundary $\partial\tau_\alpha$, are conserved (dl is an element of length of the contour $\partial\tau_\alpha$).

We will assume that exact steady solutions of problem (3.1)–(3.4), (3.7)–(3.9) exist, namely

$$\psi = \Psi(x,y), \quad \omega = \Omega(x,y), \quad q = Q(x,y) \tag{3.13}$$

where the fields Ψ, Ω and Q satisfy the relations

$$[\Psi,\Omega] = 0, \quad [\Psi,Q] = 0 \tag{3.14}$$

Equalities (3.14) correspond to the existence of the following functional relations

$$f_0(\Psi,\Omega) = 0, \quad f_1(\Psi,Q) = 0, \quad f_2(\Omega,Q) = 0$$

which, when $\nabla Q \neq 0$ in the flow [11], are solved in the form

$$\Psi = \Psi(Q), \quad \Omega = \Omega(Q), \quad Q \in (Q^-, Q^+), \tag{3.15}$$

$$Q^- = \min Q(x,y), \quad Q^+ = \max Q(x,y) \text{ in } \tau$$

We will further investigate the unsteady solutions of problem (3.1)–(3.4), (3.7)–(3.9)

$$\psi = \Psi(Q) + \varphi(x,y,t) \tag{3.16}$$

$$\omega = \Omega(Q) + \sigma(x,y,t), \quad q = Q(x,y) + \kappa(x,y,t)$$

where the fields φ, σ, κ are regarded as perturbations of the solutions (3.13) and (3.15). We will assume that, when the initial data (3.9) is appropriately specified, solutions (3.16) exist, are continuous and their derivatives in (3.1)–(3.4) and (3.7)–(3.9) are also continuous.

Suppose the fields ω and q depend on one another in accordance with one of the following two laws.

1. The fields $q(x, y, 0)$ and $\omega(x, y, 0)$ in (3.6) and (3.9) are obtained from $Q(x, y)$ and $\Omega(x, y)$ (3.13) using one field of mutual displacements of the particles of the incompressible fluid. The quantities q and ω in such displacements are constant in each fluid particle and equal to their values in the unperturbed flow (3.13). Then

$$\omega = \Omega(q), \quad q \in (Q^-, Q^+), \quad \omega \in (\Omega^-, \Omega^+) \tag{3.17}$$

$$\Omega^- = \min \Omega(x,y), \quad \Omega^+ = \max \Omega(x,y) \text{ in } \tau$$

It follows from (3.17) that the relation between q and ω and the intervals in which these quantities are defined are the same as in (3.15).

2. One of the fields $\omega(x, y, 0)$ or $q(x, y, 0)$ is assumed to be arbitrary, while the other is obtained from it using the relation $\omega = \Omega(q)$ in (3.17). The function $\Omega(q)$ when $q \in (Q^-, Q^+)$ is taken from (3.15) and is extended in an arbitrary (fairly continuous) way outside this interval along the whole of the q-axis. Limitations on this arbitrariness will arise later.

It is clear in both cases that the relation $\omega = \Omega(q)$, which is satisfied at the initial instant of time, remains true at any subsequent instant.

Condition 1 imposes a limitation on the initial vortex field, while condition 2 does not. Naturally condition 1 is a special case of condition 2.

Taking relations (3.15) into account we can construct an auxiliary function of two variables $V = V(X, Y)$

$$V = \Omega'(X)[\Psi(X) - \Psi(Y)];\quad X, Y \in (Q^-, Q^+) \tag{3.18}$$

Assertion 2. If the following inequality

$$0 \leqslant c_2^- \leqslant \partial V / \partial X \leqslant c_2^+ < +\infty \tag{3.19}$$

holds over the whole region τ of the flow with constants c_2^- and c_2^+, then at any instant the perturbations φ, κ can be estimated in terms of their initial values φ_*, κ as follows:

$$\int_\tau \left\{(\nabla\varphi)^2 + c_2^-\kappa^2\right\}dxdy \leqslant \int_\tau \left\{(\nabla\varphi_*)^2 + c_2^+\kappa_*^2\right\}dxdy \tag{3.20}$$

Proof. We will assume that condition 1 (3.17) for there to be no Lagrangian perturbations of the field q (3.8) is satisfied.

From integrals (3.10)–(3.12) of problem (3.1)–(3.4), (3.7)–(3.9) we set up the conserving functional [5, 8, 9]

$$R(\psi, q) = \frac{1}{2}\int_\tau \left\{(\nabla\psi)^2 + 2\Phi(q)\right\}dxdy + \sum_{\alpha=1}^{m} b_\alpha\Gamma_\alpha = R(\Psi, Q) + R_1 + R_2$$

$$R_1 = \sum_\alpha \left(\Phi_\alpha + b_\alpha\right)\int_{\partial\tau_\alpha} \mathbf{n}\nabla\varphi dl + \int_\tau f_q(Q, Q)\kappa dxdy$$

$$R_2 = \frac{1}{2}\int_\tau \left\{(\nabla\varphi)^2 + 2f_3(q, Q)\right\}dxdy \tag{3.21}$$

$$f_3(q, Q) = f(q, Q) - f(Q, Q) - \kappa f_q(Q, Q)$$

$$f(q, Q) = \Phi(q) - \Psi(Q)\Omega(q),\quad f_q(q, Q) = \partial f(q, Q)/\partial q$$

where b_α are arbitrary constant quantities and Ψ_α are the values of the steady stream function Ψ (3.13) on the contours $\partial\tau_\alpha$ (3.2). The arbitrary function $\Phi(q)$ and the constants b_α are chosen so that the following relations are satisfied

$$\Phi'(Q) = \Psi(Q)\Omega'(Q),\quad b_\alpha = -\Psi_\alpha \tag{3.22}$$

after which, by virtue of (3.21) it turns out that $R_1 = 0$ while the functional R_2 is independent of time. The function V, as can be seen from (3.18) and (3.22), satisfies the following relation

$$V(q, Q) = f_q(q, Q) \tag{3.23}$$

Using (3.19) an (3.23) we then obtain the inequalities

$$0 \leqslant c_2^- \leqslant f_{qq}(q, Q) \leqslant c_2^+ < +\infty \tag{3.24}$$

It follows from (3.24) that the function f is convex over the whole range of variation of the argument q (3.17). Using the expression for the residual term in Lagrangian form [13], we can write the function f_3 in (3.21) in the form

$$f_3(q, Q) = \frac{1}{2} f_{qq}(Q_*, Q)\kappa^2,\quad Q_* = Q + \vartheta\kappa \tag{3.25}$$

$$0 < \vartheta < 1,\quad \kappa = q - Q$$

Relations (3.24) and (3.25) lead to the following double inequality

$$c_2^- \kappa^2/2 \leqslant f_3(q,Q) \leqslant c_2^+ \kappa^2/2 \tag{3.26}$$

from which, taking into account the fact that R_2 of (3.21) is independent of time, the limit (3.20) also follows.

If condition 2 is satisfied, the need arises to consider the values of q outside the interval (Q^-, Q^+). For this the function $f(q, Q)$ in (3.21) is supplemented over the whole q-axis so that the truth of inequalities (3.26) is preserved. Using the two functions $\Phi(q)$ and $\Omega(q)$, that are arbitrary outside (Q^-, Q^+), it is obvious that this extension can be carried out in an infinite number of ways. As a result, from (3.26), where now $-\infty < q < +\infty$, and the conditions for the functional R_2 to be constant in time, we again obtain the required limit (3.20). This proves Assertion 2.

In the problem of the non-linear stability of plane steady flows of an ideal fluid of uniform density in a magnetic field perpendicular to the plane of motion, the sufficient condition for such flows to be stable to finite perturbations of the same symmetry was obtained in [9], which is identical with the result obtained earlier [5] and has the form

$$0 \leqslant c_4^- \leqslant d\Psi/d\Omega \leqslant c_4^+ < +\infty \tag{3.27}$$

where c_4^- and c_4^+ are constants. Comparison of (3.27) and (3.19) shows that the first is a special case of the latter if $q = \omega$, defined in (3.6).

Thus, Assertion 2 when $q \neq \omega$ gives criteria for the stability of the flows of an ideal fluid considered within the framework of new definitions, different from that proposed in [5, 9].

4. EXAMPLE

Consider the steady helical flows of an ideal incompressible fluid of uniform density in a magnetic field

$$\mathbf{u}^0 = \left(0,\ \upsilon^0,\ \frac{a\upsilon^0}{br}\right)^0,\quad \mathbf{h}^0 = \left(0,\ h_2^0, \frac{ah_2^0}{br}\right),\quad \upsilon^0 = c_5\frac{r}{R},\quad h_2^0 = c_6 r \tag{4.1}$$

where c_5 and c_6 are arbitrary constants. If we choose the coordinate r as Q, we obtain the following equations from the sufficient condition of non-linear stability (2.12) for the flows (4.1)

$$\sum_{i=1,2}\left\{-\rho_i^{0'}(r)U_i'(r)+\rho_i^{0''}(r)[U_i(r_0)-U_i(r)]\right\}=0,\quad c_1^-=0$$

In this case the limit of stability (2.13) takes the form

$$\int_\tau \left(\lambda^2 R^{-1}+u^2\right)d\tau \leqslant \int_\tau \left(\lambda_*^2 R^{-1}+u_*^2+c_1^+\kappa_*^2\right)d\tau$$

Note that the sufficient condition for non-linear stability (2.20) [6, 7] cannot be applied to the flows (4.1), since non-negative constants c_3^- and c_3^+, which would set an upper and lower limit, respectively, to the quantity $g_1/(\rho_1^0)$, do not exist

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